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BAKALÁŘSKÁ PRÁCE

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Integrální Transformace

Katedra Matematické Analýzy

Vedoucí bakalářské práce: prof. RNDr. Luboš Pick, CSc., DSc.
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Děkuji zejména panu vedoucímu práce za takřka bezbřehou trpělivost, dále svým blízkým za chápavé prostředí, Vítu Musilovi za TEXnické rady a všem přátelům, kteří pomohli a poradili. Také všem, kteří se podíleli na mém matematickém vzdělávání, zejména Mgr.Helena Kommová, RNDr.Martin Kapoun, Michal Rolínek, Matematická Olympiáda a Matematický korespondenční seminář. Též děkuji P.R.Halmosovi za knihu [8], která mi vrátila chuť do psaní práce. Díky.

Prohlašuji, že jsem svou bakalářskou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce a jejím zveřejňováním.

V Praze dne Dominik Mokriš

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Abstrakt: V předložené práci studujeme různé základní vlastnosti Hilbert – Schmidtových operátorů, zejména jejich kompaktnost. Podáváme také dvě ekvivalentní formulace Hilbert – Schmidtovy vlastnosti. Ve třetí kapitole studujeme vztah kompaktních, slabě kompaktních, prekompaktních a singulárních operátorů a následně podáváme příklady operátorů, které splňují pouze některé z těchto vlastností právě z oblasti integrálních transformací. Práce je kompilace z různých zdrojů.

Klíčová slova: Integrální, transformace, operátor

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Abstract: In the present work we study several basic properties of the Hilbert–Schmidt operators, mainly their compactness. We also give two equivalent formulations of Hilbert–Schmidt phenomenon. In the third chapter, we study relationship of compact, weakly compact, precompact and strictly singular operators and consequently we give examples of operators, which carry out only some of these peculiarities right from the area of integral transforms. The work is a compilation from several sources.

Keywords: Integral, transform, operator

Chapter 1

Introduction

1.1 Introduction and Motivation

"Nature laughs at the difficulties of integration." Pierre Simon Laplace [7]

In this thesis we study properties of various operators with special emphasis on integral transforms. Integral transforms are a subject of great both theoretical and practical importance. In applications, mainly Laplace and Fourier transform appear for their quality in simplifying differential equations. They often can be used to simplify a given problem or to replace it with a simple one (for example, they can reduce differential equations to algebraic ones). Further transforms play their roles, particularly Hilbert transform, Stieltjes transform, Mellin transform, Hankel transform, Kontorovich–Lebedev transform, Mehler–Fock transform and many others. Several applications of the Laplace transform may be found in [4] as well as applications of other transforms [2]. More rigorous introduction to Laplace transform may be found in [14] and in [10].

From the theoretical point of view, there are at least two motivations to study integral transforms. One may observe that taking several special kernels many concepts may be covered, such as the operator of primitive function by kernel $k(s,t) = \chi_{[0,s]}(t)$ or operator of convolution with g by kernel $k(s,t) = g(s-t)$. For some well–known details about convolution, see Theorem 3.5. What's behind these phenomenons is the Schwarz Kernel Theorem. For its proof and some more details, see [5].

The second theoretical motivation to study integral transforms is genuinely practical–it is a rich source of nontrivial examples. That's exactly the direction of this thesis. We illustrate this on the concept of strictly singular and compact operators. In chapter 2, we survey several important properties of a special operator class – those induced by a kernel that is in L^2 with the product measure, called Hilbert–Schmidt operators, which all turn out to be compact. Chapter 3 deals with relations between strictly singular, compact, precompact and weakly compact operators and strict inclusions are illustrated by integral operators with special kernels. We set the course of action by this diagram. Several problems are included to take a think.

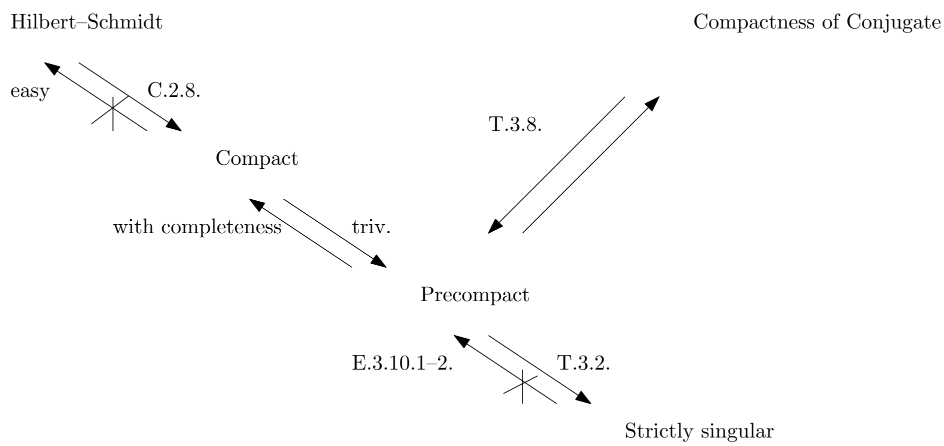

Figure 1.1: plan of work

1.2 Notation and Preliminaries

We use measure-theoretic and analytic pieces of knowledge from the first two years of study on our faculty with no warning. We also use Riesz's lemma, Arzelá–Ascoli's theorem and other findings of the Functional Analysis. Common notation of $L^p[a,b]$ for the function classes with equality almost everywhere that are integrable in the p-th power over the interval $[a,b]$, naturally with usual norms, is in use. Symbol c_0 denotes the space of all sequences tending to zero as well as l^2 space of all square-summable sequences. Symbol $\mathcal{L}(X,Y)$ denotes set of all bounded linear operators with domain in X and range in Y, where X and Y with no notification are normed linear spaces. When talking about "rank 1," we mean dimension of range of the discussed operator to be 1 (or any other number instead). Notions of compactness and the related concepts are fixed in Definition 3.1.

Useful tool is also the Hölder inequality

$$\int fg \leq \|f\|_p \|g\|_{p'}, \qquad \tfrac{1}{p} + \tfrac{1}{p'} = 1$$

and its direct consequence

$$\int_E |f|^q dx \overset{\text{Hölder}}{\leq} \left(\int_E |f|^{q \frac{p}{q}} dx \right)^{\frac{q}{p}} \left(\int_E 1^{\frac{p}{p-q}} \right)^{\frac{p-q}{p}} = \|f\|_p^q |E|^{\frac{p-q}{p}}, \tag{1.1}$$

where E denotes the entire space.

When we talk about operator K induced by a kernel k, we mean

$$Kf(s) = \int k(s,t) f(t) dt.$$

When we denote an operator by a letter in capitals, we usually use the same letter for its kernel but in lower case.

Chapter 2

L^2-theory of kernel operators

2.1 Schur test

We introduce one more tool to tackle with the task of establishing boundedness of integral operator. It is a consequence of Hölder's inequality but it is useful also at occasions, when a direct application of Hölder's inequality isn't sufficient. This tool or its Corollary is often called "Schur test."

Theorem 2.1. *Let k be a nonnegative measurable function on $X \times X$, let K be an integral operator induced by k and $1 < p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If there exists a constant $C > 0$ and a function h, that is positive and measurable on X with*

$$\int_X k(s,t)h(t)^q d\mu(t) \leq Ch(s)^q$$

for μ-almost all $x \in X$ and

$$\int_X k(s,t)h(s)^p d\mu(s) \leq Ch(t)^p$$

for μ-almost all $y \in X$, then K is bounded on $L^p(X, \mu)$ with norm not greater than C.

Proof. We use Hölder's inequality and Fubini's Theorem. If $f \in L^p(X, \mu)$, then for any $s \in X$,

$$|Kf(s)| \leq \int_X k(s,t)h(t)h^{-1}(t)|f(t)|d\mu(t),$$

and we may use Hölder's inequality to prove

$$|Kf(s)| \le \left(\int_X k(s,t)h(t)^q d\mu(t)\right)^{1/q} \left(\int_X k(s,t)h(t)^{-p}|f(t)|^p d\mu(t)\right)^{1/p}.$$

If we use the first inequality of the assumption, we have

$$|Kf(s)| \le C^{1/q}h(t)\left(\int_X k(s,t)h(t)^{-p}|f(t)|^p d\mu(t)\right)^{1/p}$$

for all $s \in X$. Applying Fubini's theorem and the second inequality in the assumption, we obtain

$$\int_X |Kf(s)|^p d\mu(s) \le C^{p/q}\int_X |f(t)|^p h(t)^{-p}\int_X k(s,t)h(s)^p d\mu(s)d\mu(t) \le$$

$$\le C^{1+p/q}\int_X |f(t)|^p d\mu(t) = C^p\int_X |f(t)|^p d\mu(t).$$

From this it is obvious that K is a bounded integral operator on $L^p(X,\mu)$ with the norm not exceeding C. □

The following results may be derived as a Corollary of Theorem 2.1 or directly, using the very same arguments.

Corollary 2.2. *If k is a nonnegative measurable function on $X \times X$ and if there is a constant $C > 0$ and a positive measurable function h on X such that*

$$\int_X k(s,t)h(t)d\mu(t) \le Ch(s)$$

for μ-almost all $s \in X$ and

$$\int_X k(s,t)h(s)d\mu(s) \le Ch(t)$$

for almost all $t \in X$, then the integral operator K induced by the kernel k is bounded on $L^2(X,\mu)$ with norm lesser than or equal to C.

The corollary may be modified using positive constants α, β instead of C in the respective equations. Also it is not necessary to use h in both functions, we only need two positive functions p, q such that

$$\int k(s,t)q(t)dt \le \alpha p(s) \tag{2.1}$$

$$\int k(s,t)p(s)ds \le \beta q(t), \tag{2.2}$$

then the kernel induces a bounded operator with

$$\|K\| \leq \sqrt{\alpha\beta}.$$

For more details and examples of applications, c.f. [8, Theorem 5.2].

2.2 Compactness of Hilbert–Schmidt operators

Throughout this chapter X and Y will denote separable normed linear spaces.

Definition 2.3. A Hilbert–Schmidt operator is an operator induced by a kernel in $L^2(X \times Y)$.

Definition 2.4. Let u and v be square integrable but not necessarily bounded functions on $L^2(S)$ and $L^2(T)$, respectively, then their `tensor product` is defined by

$$(u \otimes v)(s,t) = u(s)\overline{v(t)}$$

and it forms a square integrable function in $L^2(S \times T)$.

The following lemma and its generalising corollary describe an equivalence between special class of kernels and a special class of Hilbert–Schmidt operators. It will be used in proving the principial result of this section–Corollary 2.8.

Lemma 2.5. *If u and v are elements of $L^2(X)$ and $L^2(Y)$ respectively, both not being zeros, and if $k = u \otimes v$, that means $k(s,t) = u(s)\overline{v(t)}$, then the Hilbert–Schmidt operator K has rank 1, conversely, if K is an arbitrary integral operator with rank 1 from $L^2(X)$ to $L^2(Y)$, then there exists a kernel k of the form $u \otimes v$ for some u and v such that K is induced by k.*

Proof. While $Kg(s) = \int u(s)\overline{v(t)}g(t)dt =< g, v > u(s)$, it is obvious that every vector in the range of K is a scalar multiple of u.

If, conversely, K is an operator with rank 1 and u a fixed nonzero element in the range of K, then every Kg for $g \in L^2(Y)$ must be some multiple of u. Thus $Kg =< g, v > u$ for some $v \in L^2(Y)$,

$$Kg(s) = \int u(s)\overline{v(t)}g(t)dt$$

or, equivalently, K is induced by a kernel of the form $k = u \otimes v$. □

By induction we may derive the following corollary.

Corollary 2.6. *If $u_1, \ldots, u_n$ are elements of $L^2(X)$ and $k = \sum_{j=1}^n u_j \otimes v_j$, then rank $K \leq n$, if, conversely, K is an arbitrary bounded operator from $L^2(Y)$ to $L^2(X)$ with rank not greater than n, then K is induced by some kernel of the form $\sum_{j=1}^n u_j \otimes v_j$.*

The following theorem is often reffered to as a backbone of the whole Hilbert–Schmidt theory and is of classical importance–some authors define the Hilbert–Schmidt operators vice versa with this equal characterisation.

Theorem 2.7. *If A is a Hilbert–Schmidt operator from $L^2(T)$ to $L^2(S)$ and $\{g_j\}$ is an orthonormal set in $L^2(T)$, then*

$$\sum_j \|Ag_j\|^2 < \infty.$$

Conversely, a bounded linear operator A from $L^2(T)$ to $L^2(S)$ with $\sum_j \|Ag_j\|^2 < \infty$ for some orthonormal basis $\{g_j\}$ in $L^2(T)$ is a Hilbert–Schmidt operator.

Proof. Tensor products of the form $f \otimes g$, where f and g are linear combinations of elements of orthonormal bases $\{f_i\}$ and $\{g_j\}$ respectively, are dense in the set of all vector products $u \otimes v$ and finite linear combinations of products $u \otimes v$ are dense in $L^2(X \times Y)$ by the definition of product measure and thus vectors $f_i \otimes g_j$ form orthonormal basis in $L^2(X \times Y)$. For arbitrary bounded linear operator T we get from the Parseval's identity

$$\sum_j \|Tg_j\|^2 = \sum_i \sum_j | < Tg_j, f_i > |^2. \tag{2.3}$$

On the other hand, for $k \in L^2(X \times Y)$ we see that

$$< k, f_i \otimes g_j > = \int \int k(s,t)\overline{f_i(s)} g_j(t) ds dt = \tag{2.4}$$

$$= \int \int k(s,t) g_j(t) dt \overline{f_i(s)} ds =< Tg_j, f_i > . \tag{2.5}$$

If T is an integral operator induced by a kernel $k \in L^2(X \times Y)$, then by (2.4) and (2.5) the right hand side of (2.3) is finite–it is equal to $\|k\|_2^2$. Thus also the left hand side is finite, no matter which orthonormal basis $\{g_j\}$ has been used.

If, conversely the left hand side of the equation (2.3) is finite for some orthonormal basis $\{g_j\}$, then also the right hand side is finite, which means,

$$\sum_i \sum_j < Tg_j, f_i > (f_i \otimes g_j)$$

converges in $L^2(X \times Y)$ to some element k of $L^2(X \times Y)$ with Fourier coefficients $< k, f_j \otimes g_i >=< Tg_j, f_i >$. From this and from the equation (2.4) we get that T is an integral operator induced by k. $\square$

Corollary 2.8. *Every Hilbert–Schmidt operator is compact.*

Proof. For K use the Fourier expansion as in Theorem 2.7

$$k = \sum_i \sum_j < k, f_i \otimes g_j > (f_i \otimes g_j),$$

it shows that k is the L^2 limit of finite sums of tensor products. From the inequality $\|K\| \leq \|k\|_2$, which is obvious by Cauchy–Schwarz inequality, it follows that K is the norm limit of the integral operators induced by those finite sums. By Corollary 4.4, they have finite ranks and thus are compact. $\square$

As the next result shows, even more can be derived.

Corollary 2.9. *A bounded linear operator $T : L^2(Y) \to L^2(X)$ is a Hilbert–Schmidt operator if and only if it is compact and $\sum \lambda_j^2 < \infty$, where λ_j's are eigenvalues of $\sqrt{TT^*}$, with multiplicities counted.*

Proof. For operator $T : L^(Y) \to L^2(X)$, $\{\lambda_j\}$ family of scalars, and $\{g_j\}$ the corresponding orthonormal basis of $L^2(Y)$ with

$$\sqrt{T^*T} g_j = \lambda_j g_j$$

we have

$$\sum_j \lambda_j^2 = \sum_j < T^*Tg_j, g_j >= \sum_j \|Tg_j\|^2. \tag{2.6}$$

For T Hilbert–Schmidt the operator $\sqrt{T^*T}$ is Hermitian (self adjoint) and compact and thus, e.g. by [11, Corollary 8.20], it has an orthonormal basis of eigenvectors. Thus from (2.6) and Theorem 2.7 sum of the squares of eigenvalues is finite.

Conversely, if $\sqrt{T^*T}$ is compact with finite sum of squares of eigenvectors, then from (2.6) and Theorem 2.7 follows that T is Hilbert–Schmidt. $\square$

Problem 1. If the integral operator 0 is induced by a kernel k, then $k = 0$ for almost all x and y.

Problem 2. Identity operator on $L^2[0, 1]$ is not an integral operator

Solutions to these problems may be found in [8, Theorem 8.1 and Theorem 8.5].

Chapter 3

Strictly singular operators

3.1 Theorems and definitions

In this section, we explore some facts about strictly singular operators as well as their relationship to compact operators. These new pieces of knowledge will be applied in the next section to integral transforms.

Definition 3.1. A linear operator T from a normed linear space X to a normed linear space Y is called

- `precompact` if TS is totally bounded in Y, S means the unit sphere in X,
- `compact` if $\overline{TS}$ is compact in Y,
- `strictly singular` if it has no bounded inverse on any infinite–dimensional subspace contained in its domain.
- `weakly compact` if it takes bounded sequences onto sequences that have a weakly convergent subsequence.

Theorem 3.2. *Every precompact operator is strictly singular.*

Proof. Let X, Y be normed linear spaces and $B : X \to Y$ a precompact operator. This operator B is clearly bounded, while it takes every bounded set onto a totally bounded set, which is bounded. Suppose B has an inverse that is bounded on a subspace M of B's domain. Then BS_M is totally bounded, where S_M means the unit sphere in M. Since B has a bounded inverse on M, S_M must be totally bounded in M. Hence M is finite dimensional due to Riesz's Lemma. $\square$

The following two theorems are used in what follows but their proofs are omitted. The reader may find them in [3, Theorem VI.8.12]

Theorem 3.3. *Every weakly compact map from X or X^* into a Banach space maps sequences that are weakly compact onto sequences convergent in norm whenever X is $L^1(S, \Sigma, \mu)$ or $L^\infty(S, \Sigma, \mu)$, where (S, Σ, μ) is a positive measure space, Thus every operator from X or X^* into a Banach space, which is weakly compact, is also strictly singular.*

Theorem 3.4. *If (S, Σ, μ) is a positive measure space, Σ being the $\sigma-$algebra of all Borel subsets of S, μ measure with $\mu(S) < \infty$, then a sequence $\{y_n\}$ in $L^1(S, \Sigma, \mu)$ has a weakly convergent subsequence if and only if it is bounded in $L^1(S, \Sigma, \mu)$ and $\int_E y_n d\mu$ converges to zero uniformly for all y_n as $\mu(E) \to 0$.*

Theorem 3.5. *Given $f \in L^p(\mathbb{R}), 1 \leq p < \infty$ and $g \in L^1(\mathbb{R})$, the convolution $h = f * g$ defined for almost all t by*

$$h(t) = \int_{\mathbb{R}} f(t-s)g(t)dt$$

is in $L^p(\mathbb{R})$ and

$$\|h\|_p \leq \|f\|_p \|g\|_1.$$

To prove this, we will use the famous Riesz–Thorin theorem, which is rather classical and we will not give the proof – an interested reader may find it in [1, p. 196]. It also can be taken as an invitation to interpolation theory.

Theorem 3.6. *(Riesz–Thorin) Let $1 \leq p_0, q_0, p_1, q_1 \leq \infty, \quad 0 \leq \theta \leq 1$, let $T : L^{p_0} \to L^{q_0}$ and also $T : L^{p_1} \to L^{q_1}$, then $T : L^p \to L^q$, where*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \text{ and } \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

for some $\theta \in [0, 1]$. Moreover,

$$\|T\|_{L^p \to L^q} \leq \|T\|^{1-\theta}_{L^{p_0} \to L^{q_0}} \|T\|^{\theta}_{L^{p_1} \to L^{q_1}}.$$

Proof of Theorem 3.5. Define an operator

$$T_g : f \mapsto f * g,$$

for a fixed function $g \in L^1$. We use usual well known estimates of $f * g$:

$$\begin{aligned}
&\|f * g\|_1 \leq \|f\|_1 \|g\|_1 && \text{(Cauchy–Schwarz)},\\
&\|f * g\|_\infty \leq \|f\|_p \|g\|_{p'} \quad \forall p \in [1, \infty], \tfrac{1}{p} + \tfrac{1}{p'} = 1 && \text{(Hölder)},\\
&\|f * g\|_\infty \leq \|f\|_\infty \|g\|_1 && \text{(trivial)}.
\end{aligned}$$

That means,

$$T_g : L^1 \to L^1 \text{ with a constant } \|g\|_1,$$

$$T_g : L^\infty \to L^\infty \text{ with a constant } \|g\|_1,$$

so we may use Riesz–Thorin Theorem to get

$$T_g : L^p \to L^p \qquad \forall p \in [0, \infty] \text{ with a constant } \|g\|_1.$$

That is

$$\|f * g\|_p \leq \|f\|_p \|g\|_1,$$

what was to prove. □

Remark 3.7. We may proceed further and apply the Riesz–Thorin Theorem to get yet less trivial piece of knowledge. Define operator $S_f : g \mapsto f * g$ for a fixed function $f \in L^p$. We have that

$$S_f : L^{p'} \to L^\infty \text{ with a constant } \|f\|_p,$$

$$S_f : L^1 \to L^p \text{ with a constant } \|f\|_p$$

from the foregoing step and thus, using Riesz–Thorin Theorem, we get

$$\|f * g\|_r = \|f\|_p \|g\|_q, \text{ where } \tfrac{1}{r} + 1 = \tfrac{1}{p} + \tfrac{1}{q}.$$

Compare this to the proof given in [12, Theorem 26.20]. It is usually called "Generalised Young's inequality." When the constants are the same, it is easy. If they were different, we would have to compute θ to get the estimate.

Lemma 3.8. *If a sequence $\{K_n\}$ of precompact operators in $L(X, Y)$ satisfies $K_n \to K$ in $L(X, Y)$, then K is precompact.*

Proof. For a given $\varepsilon > 0$ there exists an integer N such that

$$\|K - K_N\| < \frac{\varepsilon}{3}.$$

Moreover, from the precompactness of K_n there exist elements $x_1, x_2, \dots, x_m$ in the unit ball S_X such that for a given x there is x_i

$$\|K_N x - K_N x_i\| < \frac{\varepsilon}{3}.$$

Using these two facts, we see that

$$\|Kx - Kx_i\| \leq \|Kx - K_N x\| + \|K_N x - K_N x_i\| + \|K_N x_i - K x_i\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3},$$

which implies that K is precompact. $\square$

Theorem 3.9. *A bounded linear operator is precompact if and only if its conjugate is compact.*

Proof. Let $K : X \to Y$ be linear bounded and precompact. We will show that K' is precompact and therefore compact, while X^* is complete due to the completeness of scalars. For $\varepsilon > 0$ there exist $x_1, \dots, x_n$ in the unit sphere S_X such that for each $x \in S_X$ there exist an x_i with

$$\|Kx - Kx_i\| < \varepsilon/3. \tag{3.1}$$

If T is a bounded map from Y^* defined as

$$Ay' = (y'Kx_1, \dots, y'Kx_n),$$

then it is compact – clearly it is bounded and linear operator with range in a finite dimensional space. Hence there exist $y'_1, \dots, y'_m$ in the unit ball $S_{Y'}$ such that for $y' \in S_{Y'}$ there exists an y'_j such that

$$\|Ay' - Ay'_j\| < \varepsilon/3.$$

In particular

$$|y'Kx_i - y'_j Kx_i| < \varepsilon/3, \qquad 1 \leq i \leq n. \tag{3.2}$$

Using equations (3.1) and (3.2) we may derive that

$$\begin{aligned} |K'y'x - K'y'_j x| &\leq |y'Kx - y'Kx_i| + |y'Kx_i - y'_j Kx_i| + |y'_j Kx_i - y'_j Kx| \\ &\leq \|Kx - Kx_i\| + \varepsilon/3 + \|Kx_i - Kx\| < \varepsilon. \end{aligned}$$

Thus

$$\|K'y' - K'y'_j\| \leq \varepsilon$$

and therefore K' is compact.

Conversely, for K' compact, by what has just been derived, K'' is compact. Now, for J_X, J_Y natural maps from X to X^{**} and Y into Y^{**} respectively, $K''J_X = J_Y K$. Left hand side, i.e. $K''J_X$, is compact because of compactness of K'' and boundedness of J_X and thus, in particular, $J_Y K$ is precompact. It is easy to see that K is precompact, while J_Y has a bounded inverse, which now completes the whole proof. $\square$

Example 3.9.1. Compactness of the conjugate of a bounded linear operator does NOT imply compactness of the operator, as this example shows.

Let $T_0 : c_0 \to l^2$ be defined by

$$T_0(\{\alpha_k\}) = \{\frac{\alpha_k}{k}\}, \qquad 1 \leq k.$$

Define now T by the same formula but take it as an operator onto the range of T_0, denoted as Y. Then the T is not compact.

To prove that, let $x_k = \{1, 1, \dots, 1, 0, 0, \dots\}$, where the number of 1's is exactly the k. Then $\|x_k\|_{c_o} = 1$ but $Tx_k \to y = \{1, 1/2, 1/3, \dots\}$ in l^2. But y cannot be in Y, because $Tx = y$ would imply $x = \{1, 1, \dots\}$, which is not in c_0. Thus $\{Tx_k\}$ can't have any subsequence convergent in Y, i.e. T is not compact.

However, $T_n : c_0 \to Y$ defined by $T(\{\alpha_n\}) = \beta_n$ for β_n being α_k/k for the first n occasions and zero else, converge in $\mathcal{L}(c_0, Y)$ to T. And what more–ranges of $T_n's$ are finite dimensional and thus T_n's are compact.

We now know that $T_n \to T$ in $\mathcal{L}(c_0, Y)$, thus $T_n' \to T'$ in $\mathcal{L}(Y^*, c_0^*)$. Since every T_n is compact, so is T_n'. Hence T' is precompact as a limit of precompact operators and compact thanks to completeness of Y^*. But T is not compact.

Remark 3.10. From the Functional Analysis we know that every bounded sequence in a reflexive space has a weakly convergent subsequence. Thus if at least one from X and Y is reflexive, then every operator in $\mathcal{L}(X, Y)$ is weakly compact. Using this and Theorem 3.3 we gain that if X is one of the spaces in Theorem 3.3, then every linear bounded map from X or X^* into a reflexive space Y is strictly singular.

Problem 3. Does every bounded linear operator on an infinite–dimensional separable Hilbert space have a nontrivial invariant subspace?

3.2 Application to integral transforms

Lemma 3.11. *If k is a measurable kernel on $[0,1]\times[0,1]$ inducing integral operator K by*

$$(Kf)(s)=\int_0^1 k(s,t)f(t)dt$$

as a map from $L^1[0,1]$ into $L^p[0,1]$, then K is weakly compact.

Proof. For a measurable subset E of $[0,1]$ and a function $f\in L^1[0,1]$ we see that

$$\left(\int |(Kf)(s)|^p dt\right)^{1/p} \le (\mu(E))^{1/p} \sup_{0\le s,t\le 1} |k(s,t)| \int_0^1 |f(t)|dt. \tag{3.3}$$

Considering $E=[0,1]$ and $1<p<\infty$, the equality (3.3) shows K is bounded and therefore weakly compact by Remark 3.10.

For $p=1$ let $\{f_n\}$ be a bounded sequence in $L^1[0,1]$, then, by 3.3 $\{Kf_n\}$ is bounded in $L^1[0,1]$ and $\int_E |(Kf_n)(t)|dt$ converges to zero uniformly for all f_n as $\mu(E)\to 0$. From Theorem 3.4 it follows that K is weakly compact. $\square$

The next two examples show that Theorem 3.2 cannot be reversed. That means, they give examples of strictly singular but not compact operators.

Example 3.11.1. For $n=1,2,\dots$ let k be defined on $[0,1]\times[0,1]$ by

$$k(s,t)=\begin{cases} 0 & (s,t)\in\left(\frac{2j}{2^n},\frac{2j+1}{2^n}\right]\times\left(\frac{1}{2^n},\frac{1}{2^{n-1}}\right], j=0,1,\dots,2^{n-1}-1,\\ 2 & (s,t)\in\left(\frac{2j+1}{2^n},\frac{2j+2}{2^n}\right]\times\left(\frac{1}{2^n},\frac{1}{2^{n-1}}\right], j=0,1,\dots,2^{n-1}-1,\\ 0 & \text{when } s=0 \text{ or } t=0.\end{cases}$$

Kernel k is clearly bounded and measurable. The induced integral operator $K: L^1[0,1]\to L^p[0,1], 1\le p<\infty$ defined by

$$(Kf)(s)=\int_0^1 k(s,t)f(t)dt$$

is strictly singular by Lemma 3.11 and Theorem 3.3 but it is not compact. To see this, define a sequence $\{f_n\}$ by

$$f_n(t)=\begin{cases} 2^n & t\in\left(\frac{1}{2^n},\frac{1}{2^{n-1}}\right], n=1,2,\dots\\ 0 & \text{otherwise}\end{cases}.$$

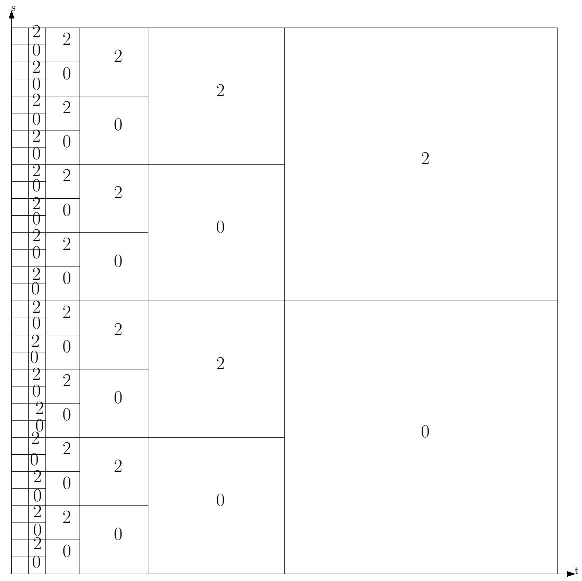

Figure 3.1: The kernel k, familiarly called "pedigree"

Then $\|f_n\|_1 = 1$ and we can calculate

$$\|Kf_n - Kf_m\|_p \geq \|Kf_n - Kf_m\|_1 = 1, n \neq m,$$

where $\|.\|_p$ means the L^p norm in $L^p[0,1]$, hence $\{Kf_n\}$ can't have any convergent subsequence in $L^p[0,1]$.

Example 3.11.2. Let k be bounded and measurable kernel on $[0,1] \times [0,1]$ and let φ be in $L^1[-1,1]$. From Theorem 3.5, operator K, defined by

$$(Kf)(s) = \int_0^1 k(s,t)\varphi(s-t)f(t)dt,$$

is a bounded mapping from $L^1[0,1]$ into $L^1[0,1]$. We will show that K is weakly compact and therefore, according to Theorem 3.3, it is strictly singular. Denote $[0,1]$ as I.

Let E be a measurable subset of I. Then for $f \in L^1[0,1]$ and $M = \sup_{0\leq s,t\leq 1} |k(s,t)|$ Fubini's theorem says that

$$\int_E |(Kf)(s)ds \leq M \int_E ds \int |\varphi(s-t)||f(t)|dt =$$
$$= M \int_I \int_{E-t} |\varphi(s)ds||f(t)|dt.$$
$$\varphi \in L^1[-1,1] \Rightarrow \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall E, |E| < \delta :$$
$$\int_{E-t} |\varphi(s)|ds < \varepsilon, t \in I, \Rightarrow \int_E |(Kf)(s)|ds \leq M\varepsilon, \|f\| \leq 1.$$

Thus K is weakly compact due to Theorem 3.4.

Remark 3.12. Let's now have a look at a particular case. If g is bounded and measurable on $[0,1] \times [0,1]$, the operator K_1, defined by

$$(K_1 f)(s) = \int_0^1 \frac{g(s,t)}{|s-t|^a} f(t)dt, \quad 0 < a < 1,$$

is strictly singular as a map from $L^1[0,1]$ to $L^1[0,1]$. Taking

$$g(s,t) = k(s,t)|s-t|^a, \quad 0 < a < 1,$$

where k is given in lemma 3.11, $K_1 = K$, which is not compact.

Lemma 3.13. *Let $1 < p, q < \infty$ and p', q' be the conjugates of p, q respectively, suppose $k(s,t)$ is in $L^r[0,1] \times [0,1]$, where $r = \max\{p', q\}$, then the linear operator K defined by*

$$(Kf)(s) = \int_0^1 k(s,t)f(t)dt$$

is a compact map from $L^p[0,1]$ to $L^q[0,1]$.[1]

Proof. First, we need to show that K is bounded and from $L^p[0,1]$ to $L^q[0,1]$. Observe

$$\|Kf\|_q = \left(\int_0^1 \left|\int_0^1 k(s,t)f(t)dt\right|^q ds\right)^{1/q} \overset{\text{Hölder}}{\leq}$$
$$\leq \left(\int_0^1 \left(\int_0^1 |k(s,t)|^r dt\right)^{q/r} \|f\|_{r'}^q ds\right)^{1/q} \overset{1.}{\leq}$$

[1] Notice that we are not assuming any relationship between p and q

$$\leq \|f\|_p^{q/q} \left(\int_0^1 \left(\int_0^1 |k(s,t)|^r dt \right)^{q/r} ds \right)^{1/q} \overset{2.}{\leq}$$

$$\|f\|_p \left(\int_0^1 \int_0^1 |k(s,t)|^r dt ds \right)^{q/q}.$$

<u>Explanatory notes</u>

1. Since $r' \leq p$ we see, by (1.1), that $f \in L^{r'}[0,1]$ and $\|f\|_{r'} \leq \|f\|_p$.

2. Since $0 < q/r \leq 1$, we can use Minkowski inequality or (1.1)–for any $g \in L^1[0,1]$
$$\int_0^1 |g(t)|^{q/r} dt \leq \left(\int_0^1 |g(t)| dt \right)^{q/r}.$$

Now, suppose that k is continuous on $[0,1] \times [0,1]$, we will show, K is then compact. Later, we will extend to all k's. First, take a bounded sequence $\{x_n\} \in L^p[0,1]$. We know that $\|x_n\|_1 \leq \|x_n\|_p$, therefore for $y_n = Kx_n$:

$$|y_n(s)| \leq \int_0^1 |k(s,t)x_n(t)| dt \leq \max_{0 \leq s,t \leq 1} |k(s,t)| \|x_n\|_p$$

and thus $\{y_n\}$ is uniformly bounded on $[0,1]$. Since is k is on $[0,1] \times [0,1]$ uniformly continuous,

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall t \in [0,1] : |k(s_1,t) - k(s_2,t)| < \varepsilon, \text{ whenever } |s_1 - s_2| < \delta.$$

Since $\{x_n\}$ is bounded in $L^p[0,1]$, we have for such s_1, s_2

$$|y_n(s_1) - y_n(s_2)| \leq \int_0^1 |k(s_1,t) - k(s_2,t)| |x_n(t)| dt \leq \varepsilon \|x_n\|_p$$

which shows that $\{y_n\}$ is equicontinuous on $[0,1]$. We may then use the Arzelá–Ascoli theorem. The sequence $\{y_n\}$ then contains a subsequence that is convergent in $C[0,1]$ and thus also in $L^p[0,1]$. Consequently K is compact.

Last step is the extension to kernels, which are not necessarily continuous on $[0,1]$. From the Functional Analysis we know that $C_0^\infty[0,1]$ is dense in $L^p[0,1]$ and therefore there exists a sequence $\{k_n\}$ of functions continuous on $[0,1] \times [0,1] \to k$ in $L^1[0,1]$. Define

$$K_n \in \mathcal{L}(L^p[0,1], L^q[0,1]), \qquad (K_n x)(s) := \int_0^1 k_n(s,t) x(t) dt,$$

K_n are compact according to the foregoing step. Using similar estimates as in the first step, we see that

$$\|(K_n - K)x\|_q \leq \|x\|_p \left(\int_0^1 \int_0^1 |k_n(s,t) - k(s,t)|^r ds dt \right)^{1/r} \to 0$$

as $n \to \infty$ and thus $K_n \to K$ in $L(L^p[0,1], L^q[0,1])$. The resulting K is compact due to Lemma 3.8. The proof is now complete. $\square$

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